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NEW YORK UNIVERSITY
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The Discriminant of Hill's Equation

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ABSTRACT*

Structure theorems for the discriminant, $\Delta(\omega^2)$, of Hill's differential equation are developed by means of interpolatory function theory (cardinal series representations). A method of solution of Hill's equation is developed yielding an asymptotic expansion of the discriminant for large $|\omega|$ with error term $O(|\omega|^{-9})$. Asymptotic expansions for the eigenvalues λ_n, λ'_n for large n are obtained with error terms $O(n^{-7})$. Relations between the occurrence of double zeros of the discriminant and the period of the coefficient function in Hill's differential equation are established. A discriminant-like function $D(\omega^2)$ is introduced and an interpolatory function-theoretic structure result is obtained.

*The research reported in this paper constitutes the doctoral dissertation of Mr. Jagerman. It was carried out under the direction of Professor Wilhelm Magnus.

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I. Introduction

The differential equation

$$y''(x) + \{\lambda + g(x)\}\, y(x) = 0, \tag{1.1}$$

in which $g(x)$ is a bounded periodic function of period π and mean value zero, is called Hill's equation. For a discussion of the general theory, see Reference 1, and for an investigation of the discriminant, see Reference 2. Let $y_1(x)$, $y_2(x)$ be the fundamental set of solutions of Equation (1.1) defined by

$$y_1(o) = 1, \qquad y'_1(o) = 0, \tag{1.2}$$

$$y_2(o) = 0, \qquad y'_2(o) = 1,$$

then the descriminant is defined by

$$\Delta(\lambda) = y_1(\pi) + y'_2(\pi). \tag{1.3}$$

Often it is convenient to set $\lambda = \omega^2$ so that $\Delta(\lambda) = \Delta(\omega^2)$.

The paper consists of six parts. Part II develops two general theorems concerning the well-known cardinal series of interpolation. The Paley-Wiener theorem on the Fourier integral representation of functions whose Fourier transforms vanish outside of a finite interval is established on the basis of Theorem I of the cardinal series interpolation. An explicit expression is given for the Fourier transform. These theorems enable the development of explicit representations for the discriminant in Part III. In Part IV solutions asymptotic for large $|\omega|$ of Equation (1.1) are developed. Hill's equation may be rewritten as a Riccati equation. By means of a perturbation procedure applied to the Riccati equation, an approximate solution is obtained. This approximate solution is then employed to suggest certain changes of the dependent and independent variables to effect a basic transformation of Equation (1.1). The transformed equation is then solved by rewriting it as an integral equation and employing the process of successive approximations. The solutions thus obtained are asymptotic in ω. An explicit asymptotic development of the discriminant is given with an error term of $O(|\omega|^{-9})$.

The infinitely many values of λ for which $\Delta(\lambda)^2 = 4$ are all real and constitute the set of eigenvalues of Equation (1.1). The differential equation will have solutions of period π only for those values of λ for which $\Delta(\lambda) = 2$, and solutions of period 2π only for those λ for which $\Delta(\lambda) = -2$. In Part V certain theorems are proved concerning the form of the discriminant which, together with the explicit asymptotic development found in Part IV permit the determination of the asymptotic expansions of the eigenvalues. The first four terms of the expansions are given with an error term $O(n^{-7})$.

Part VI considers the question of double zeros of $\Delta(\omega^2) + 2$ and the corresponding conditions that $g(x)$ be of period $\pi/2$. Two theorems are proved, one of which is established on the basis of an interpolation procedure providing a representation for a certain discriminant-like function $D(\omega^2)$.

II. The Cardinal Series

The cardinal series[3] is a series of the form

$$\sum_{j=-\infty}^{\infty} a_j \frac{\sin \pi(z-j)}{\pi(z-j)} . \tag{2.1}$$

Interpolation by means of the cardinal series is provided by Theorems I and II below.

Theorem I. $f(z)$ is entire ($z = x + iy$),

$$A(y) = \max_{-\infty < x < \infty} |f(x + iy)|$$

$$A(y) = o(e^{\pi/h\,|y|}),\quad h > 0,\ |y| \to \infty,$$

$$\Leftrightarrow\quad f(z) = \sum_{j=-\infty}^{\infty} f(jh) \frac{\sin \pi/h\ (z-jh)}{\pi/h\ (z-jh)}$$

The cardinal series converges uniformly in every bounded closed domain of the z-plane.

Theorem II. $f(z)$ is entire

$$A(y) = \max_{-\infty < x < \infty} |f(x + iy)|,$$

$$A(y) = o(e^{2\pi/h\,|y|}),\quad h > 0,\ |y| \to \infty,$$

$$\Leftrightarrow\quad f(z) = \sum_{j=-\infty}^{\infty} \left[\frac{\sin \pi/h\ (z-jh)}{\pi/h\ (z-jh)}\right]^2 [f(jh) + (z-jh)f'(jh)].$$

The cardinal series converges uniformly in every bounded closed domain of the z-plane.

The proof of Theorem I employs the fundamental function $\sin \pi\zeta$ while that of Theorem II employs $\sin^2 \pi\zeta$, otherwise, the proofs are completely parallel. Further, it is clearly necessary to prove Theorem I only under the supposition $h = 1$ since the extension of the theorem to interpolation at the points jh follows on replacing z by z/h.

Proof. Consider the integral

$$I^{(N)} = 1/2\pi i \int_{C_N} \frac{f(\zeta)}{(\zeta - z)\sin \pi\zeta}\, d\zeta, \tag{2.2}$$

in which C_N denotes the Nth of a sequence of paths which are squares in the ζ-plane ($\zeta = \xi + i\eta$) and whose corners are $(N + 1/2)(\pm 1 \pm i)$. For the implication to the right, it will be sufficient to prove that under the conditions of the theorem

$$\lim_{N\to\infty} I^{(N)} = 0. \tag{2.3}$$

Since the singularities of the integrand occur at $\zeta = z$, $\zeta = \pm j$ (j integral or zero), the calculus of residues yields immediately

$$\frac{f(z)}{\sin \pi z} + \sum_{j=-\infty}^{\infty} (-1)^j \frac{f(j)}{\pi(j-z)} = 0 \tag{2.4}$$

from which the cardinal series representation of $f(z)$ follows. Let

$$I^{(N)} = I_1 + I_2 + I_3 + I_4 \tag{2.5}$$

in which

$$I_1 = \frac{1}{2\pi i}\int_{-N-\frac{1}{2}}^{N+\frac{1}{2}} \frac{f(\zeta)}{(\zeta - z)\sin \pi\zeta}\, i\,d\eta, \quad \xi = N + \tfrac{1}{2}, \tag{2.6}$$

$$I_2 = \frac{1}{2\pi i}\int_{N+\frac{1}{2}}^{-N-\frac{1}{2}} \frac{f(\zeta)}{(\zeta - z)\sin \pi\zeta}\, d\xi, \quad \eta = N + \tfrac{1}{2},$$

$$I_3 = \frac{1}{2\pi i}\int_{N+\frac{1}{2}}^{-N-\frac{1}{2}} \frac{f(\zeta)}{(\zeta - z)\sin \pi\zeta}\, i\,d\eta, \quad \xi = -N - \tfrac{1}{2},$$

$$I_4 = \frac{1}{2\pi i}\int_{-N-\frac{1}{2}}^{N+\frac{1}{2}} \frac{f(\zeta)}{(\zeta-z)\sin\pi\zeta}\,d\xi, \quad \eta = -N-\tfrac{1}{2}.$$

Consider

$$I_1 = \frac{1}{2\pi i}\int_{-N-\frac{1}{2}}^{N+\frac{1}{2}} \frac{f(N+\frac{1}{2}+i\eta)}{(N+\frac{1}{2}+i\eta-z)\quad \sin\,(N+\frac{1}{2}+i\eta)}\, i d\eta\,. \tag{2.7}$$

One has

$$|I_1| \leq \frac{1}{2\pi}\int_{-N-\frac{1}{2}}^{N+\frac{1}{2}} \frac{A(\eta)}{|N+\frac{1}{2}+i\eta-z|\,|\sin\pi(N+\frac{1}{2}+i\eta)|}\,d\eta, \tag{2.8}$$

$$|N+\tfrac{1}{2}+i\eta-z| \geq \left||N+\tfrac{1}{2}+i\eta| - |z|\right| \geq \left|N+\tfrac{1}{2} - |z|\right|.$$

Choose N so large that $N+\frac{1}{2} > |z|$, then

$$|I_1| \leq \frac{1}{2\pi}\,\frac{1}{N+\frac{1}{2}-|z|}\int_{-N-\frac{1}{2}}^{N+\frac{1}{2}} \frac{A(\eta)}{|\sin\pi(N+\frac{1}{2}+i\eta)|}\,d\eta,$$

also

$$|\sin\pi\,(N+\tfrac{1}{2}+i\eta)| = \frac{e^{\pi\eta}+e^{-\pi\eta}}{2} \geq \tfrac{1}{2}e^{\pi|\eta|}, \tag{2.9}$$

hence

$$|I_1| \leq \frac{1}{\pi}\,\frac{1}{N+\frac{1}{2}-|z|}\int_{-N-\frac{1}{2}}^{N+\frac{1}{2}} e^{-\pi|\eta|}A(\eta)\,d\eta. \tag{2.10}$$

The following well-known lemma will now be employed.

Lemma. $f(x)$ bounded for all x, $\lim\limits_{x\to\infty} f(x) = 0 \implies$

$$\lim_{\lambda\to\infty}\frac{1}{\lambda}\int_0^{\lambda} f(x)\,dx = 0.$$

Proof. One has

$$\frac{1}{\lambda}\int_0^\lambda f(x)dx = \frac{1}{\lambda}\int_0^\mu f(x)dx + \frac{1}{\lambda}\int_\mu^\lambda f(x)dx,\ 0 < \mu < \lambda. \tag{2.11}$$

Choose μ so large that $|f(x)| \leq \varepsilon/2$ ($\varepsilon > 0$) for $x \geq \mu$, then

$$\left|\frac{1}{\lambda}\int_\mu^\lambda f(x)\,dx\right| < \varepsilon/2. \tag{2.12}$$

Since $f(x)$ is bounded, one may choose λ so large that

$$\left|\frac{1}{\lambda}\int_0^\mu f(x)\,dx\right| < \varepsilon/2, \tag{2.13}$$

and, hence, the lemma follows. Applying the above lemma to Equation (2.10), and using the condition $A(\eta) = o(e^{\pi|\eta|})$ of the theorem, one has

$$\lim_{N\to\infty} I_1 = 0. \tag{2.14}$$

The investigation of the integral I_3 follows exactly the same pattern as for I_1 above, hence, one also has

$$\lim_{N\to\infty} I_3 = 0. \tag{2.15}$$

The integral I_2 is given by

$$I_2 = -\frac{1}{2\pi i}\int_{-N-\frac{1}{2}}^{N+\frac{1}{2}} \frac{f(\xi + i(N+\frac{1}{2}))}{\{\xi + i(N+\frac{1}{2}) - z\}\sin \pi\{\xi + i(N+\frac{1}{2})\}}\,d\xi. \tag{2.16}$$

One has

$$|\xi + i(N+\tfrac{1}{2}) - z| \geq |\xi + i(N+\tfrac{1}{2})| - |z| \geq N+\tfrac{1}{2} - |z|\,, \tag{2.17}$$

hence

$$|I_2| \leq \frac{1}{2\pi}\frac{1}{N+\frac{1}{2} - |z|}\int_{-N-\frac{1}{2}}^{N+\frac{1}{2}} \frac{|f(\xi + i(N+\frac{1}{2}))|}{|\sin \pi(\xi + i(N+\frac{1}{2}))|}\,d\xi. \tag{2.18}$$

Also

$$|\sin \pi(\xi + i(N+\tfrac{1}{2}))| \geq \tfrac{1}{2}(e^{\pi(N+\frac{1}{2})} - e^{-\pi(N+\frac{1}{2})}) = \tag{2.19}$$

$\frac{1}{2}e^{\pi(N+\frac{1}{2})}(1 - e^{-\pi(2N+1)})$; $1 - e^{-\pi(2N+1)} \geq 1 - e^{-\pi} > 0$. Hence

$$|\sin \pi(\xi + i(N+\tfrac{1}{2}))| \geq \frac{1-e^{-\pi}}{2} \quad e^{\pi(N+\frac{1}{2})} . \tag{2.20}$$

The integral I_2 is now bounded by

$$|I_2| \leq \frac{1}{\pi(1-e^{-\pi})} \, e^{-\pi(N+\frac{1}{2})} \, A(N+\tfrac{1}{2}) \, \frac{2N+1}{N+\frac{1}{2} - |z|} . \tag{2.21}$$

Thus, by the conditions of the theorem,

$$\lim_{N \to \infty} I_2 = 0. \tag{2.22}$$

The investigation of the integral I_4 does not essentially differ from the above investigation of I_2 so that one has also

$$\lim_{N \to \infty} I_4 = 0. \tag{2.23}$$

The implication now follows. For the implication to the left, let

$$s_n = \sum_{j=-n}^{n} a_j \frac{\sin \pi(z-j)}{\pi(z-j)} . \tag{2.24}$$

The s_n are entire functions, and, since the uniform limit of a sequence of entire functions is entire, it follows that

$$f(z) = \lim_{n \to \infty} \sum_{j=-n}^{n} a_j \frac{\sin \pi(z-j)}{\pi(z-j)} = \sum_{j=-\infty}^{\infty} a_j \frac{\sin \pi(z-j)}{\pi(z-j)} \tag{2.25}$$

is entire. It is clear that

$$s_n = O(|z|^{-1} e^{\pi|y|}) = o(e^{\pi|y|}), \tag{2.26}$$

and hence,

$$f(z) = o(e^{\pi|y|}). \tag{2.27}$$

In Equation (2.25) set $z = \tau$ (τ integral), then

$$a_\tau = f(\tau), \tag{2.28}$$

hence,

$$f(z) = \sum_{j=-\infty}^{\infty} f(j) \frac{\sin \pi(z-j)}{\pi(z-j)} . \tag{2.29}$$

Thus the implication to the left follows. The sequence of functions $\sin \pi(\omega-j)/\pi(\omega-j)$ forms an ortho-normal set over $(-\infty < \omega < \infty)$. The proof follows immediately on application of the Parseval theorem for Fourier integrals[5]. Since

$$\frac{\sin \pi(\omega-j)}{\pi(\omega-j)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{iu(\omega-j)} \, du, \tag{2.30}$$

it follows that $1/\sqrt{2\pi}\, e^{iuj}$ $(|u| \leq \pi)$ is the Fourier transform of $\sin \pi(\omega-j)/\pi(\omega-j)$. Hence,

$$\int_{-\infty}^{\infty} \frac{\sin \pi(\omega-j)}{\pi(\omega-j)} \frac{\sin \pi(\omega-k)}{\pi(\omega-j)} \, d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{iu(k-j)} \, du, \tag{2.31}$$

$$= 0, \; k \neq j,$$

$$= 1, \; k = j.$$

Consider the closure in L^2 norm of the set of functions $\left\{\frac{\sin \pi(\omega-j)}{\pi(\omega-j)}\right\}_{-\infty}^{\infty}$. It follows from Theorem I that if $f(\omega)$ is entire, $f(\omega) = o(e^{\pi|\mu|})(\omega = \alpha + i\mu)$, and $f(\alpha)\epsilon L^2(-\infty,\infty)$, then $f(\omega)$ is in the closure.

The well-known theorem of Paley-Wiener may now be established.
Theorem III. $f(\omega)$ is entire $(\omega = \alpha + i\mu)$, $f(\omega) = o(e^{\pi|\mu|})$, $f(\alpha)\epsilon L^2(-\infty,\infty)$

$$\Leftrightarrow \; \exists \, F(u)\epsilon L^2(-\pi,\pi) \; \ni \; f(\omega) = \int_{-\pi}^{\pi} e^{i\omega u} F(u) du, \text{ and, in fact,}$$

$$F(u) = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} f(j) \, e^{-iuj} .$$

Proof. By Theorem I

$$f(\omega) = \sum_{j=-\infty}^{\infty} f(j) \frac{\sin \pi(\omega-j)}{\pi(\omega-j)} , \tag{2.32}$$

and, since $f(\omega)$ is in the closure of

$$\left\{\frac{\sin \pi(\omega-j)}{\pi(\omega-j)}\right\}_{-\infty}^{\infty} ,$$

one also has

$$\sum_{j=-\infty}^{\infty} |f(j)|^2 < \infty . \tag{2.33}$$

Direct substitution of the expression for F(u) given in the theorem into the Fourier intetral shows its formal validity, thus it is only necessary to show that

$$\lim_{\substack{M\to\infty \\ N\to\infty}} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\omega u} \sum_{j=M}^{N} f(j)e^{-iuj}\, du = 0. \tag{2.34}$$

Application of the Schwartz inequality yields

$$\left| \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\omega u} \sum_{j=M}^{N} f(j)e^{-iuj}\, du \right|^2 \tag{2.35}$$

$$\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{j=M}^{N} f(j)e^{-iuj} \right|^2 du.$$

Thus

$$\left| \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\omega u} \sum_{j=M}^{N} f(j)e^{-iuj}\, du \right|^2 \leq \sum_{j=M}^{N} \left|f(j)\right|^2 \to 0. \tag{2.36}$$

The limit zero is obtained on reference to Equation (2.33). By the same argument one also has

$$\lim_{\substack{M\to\infty \\ N\to\infty}} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\omega u} \sum_{j=-M}^{-N} f(j)e^{-iuj}\, du = 0, \tag{2.37}$$

and hence the implication to the right is established. For the implication to the left, consider

$$f(\omega) = \int_{-\pi}^{\pi} e^{i\omega u} F(u)\, du \tag{2.38}$$

in which $F(u)\varepsilon L^2(-\pi,\pi)$. Because of the finite interval of integration, $f(\omega)$ is clearly entire. Also since $F(u)\varepsilon L^2(-\pi,\pi)$ one has, by Plancherel's theorem, that $f(\alpha)\varepsilon L^2(-\infty,\infty)$. It remains only to show that $f(\omega)=o(e^{\pi|\mu|})$. However, the Schwartz inequality applied to Equation (2.38) immediately yields $f(\omega) = O(|\mu|^{-1/2}\, e^{\pi|\mu|})$, and hence the implication is established.

III. Analytic Properties of the Discriminant and Cardinal Series Representations

Theorem IV. The discriminant $\Delta(\omega^2)(\omega = \alpha + i\mu)$ is an entire even function of ω satisfying $\Delta(\omega^2) = O(e^{\pi|\mu|})$ for all large $|\omega|$.

A theorem of Magnus[1] and Bernstein's inequality[6] immediately yield Theorem IV. However, it is easy to prove the above results directly.

Proof. Equation (1.1) may be written in the following integral form.

$$y(x) = y(0)\cos\omega x + y^1(0)\frac{\sin\omega x}{\omega} \tag{3.1}$$

$$= \frac{1}{\omega}\int_0^x \sin\omega(x-u)g(u)y(u)\,du.$$

The functions $y_1(x)$, $y_2(x)$, therefore, satisfy

$$y_1(x) = \cos\omega x - \frac{1}{\omega}\int_0^x \sin\omega(x-u)g(u)y_1(u)\,du, \tag{3.2}$$

$$y_2(x) = \frac{\sin\omega x}{\omega} - \frac{1}{\omega}\int_0^x \sin\omega(x-u)g(u)y_2(u)\,du. \tag{3.3}$$

Since

$$|\cos\omega x| \le e^{x|\mu|}, \tag{3.4}$$

$$|\sin\omega x| \le e^{x|\mu|},$$

one has

$$|y_1(x)| \le e^{x|\mu|} + \frac{M}{|\omega|}\int_0^x e^{|\mu|(x-u)}|y_1(u)|\,du, \tag{3.5}$$

$$|y_2(x)| \le 1/|\omega|\, e^{x|\mu|} + \frac{M}{|\omega|}\int_0^x e^{|\mu|(x-u)}|y_2(u)|\,du, \tag{3.6}$$

in which M is a bound on $g(x)$. Thus,

$$g(x) \leq M \quad \text{for all } x. \tag{3.7}$$

The following lemma will now be employed.

Lemma.

$$|y(x)| \leq Ce^{x\gamma} + E \int_0^x e^{\gamma(x-u)} |y(u)| \, du$$

$$C > 0, \ \gamma > 0, \ E > 0$$

$$\Rightarrow |y(x)| \leq Ce^{(E+\gamma)x}.$$

Proof. Let

$$W = E \int_0^x e^{\gamma(x-u)} | y(u)| \, du, \tag{3.8}$$

then

$$|y(x)| \leq Ce^{x\gamma} + W, \tag{3.9}$$

$$W' = E|y(x)| + \gamma W \leq CEe^{x\gamma} + (E + \gamma)W, \tag{3.10}$$

$$W(0) = 0. \tag{3.11}$$

One now has

$$\frac{d}{dx} [e^{-(E+\gamma)x} W] \leq CEe^{-Ex}, \tag{3.12}$$

and, hence,

$$W \leq Ce^{(E+\gamma)x} - Ce^{\gamma x}, \tag{3.13}$$

The lemma follows from Equations (3.9) and (3.13). The above lemma now yields

$$|y_1(x)| \leq e^{(M/|\omega| + |\mu|)x}, \tag{3.14}$$

$$|y_2(x)| \leq \frac{1}{|\omega|} \cdot e^{(M/|\omega| + |\mu|)x}. \tag{3.15}$$

From Equation (3.3) one obtains

$$y'_2(x) = \cos \omega x - \int_0^x \cos \omega(x-u) g(u) y_2(u) \, du. \tag{3.16}$$

Hence,

$$|y'_2(x)| \leq e^{x|\mu|} + M \int_0^x e^{|\mu|(x-u)} |y_2(u)|\, du. \tag{3.17}$$

Equation (3.15) now yields

$$|y'_2(x)| \leq e^{(M/|\omega| + |\mu|)x}. \tag{3.18}$$

Thus the discriminant satisfies

$$|\Delta(\omega^2)| \leq 2e^{\pi(M/|\omega| + |\mu|)} \tag{3.19}$$

and Theorem IV is established. Since, from Equation (31.4)

$$\left| \frac{1}{\omega} \int_0^x \sin \omega(x-u) g(u) y_1(u)\, du \right| \leq \frac{M}{|\omega|} x e^{(M/|\omega| + |\mu|)x}, \tag{3.20}$$

and from Equation (3.15)

$$\left|\int_0^x \cos \omega(x-u) g(u) y_2(u)\, du\right| \leq \frac{M}{|\omega|} x e^{(M/|\omega| + |\mu|)x}, \tag{3.21}$$

one has

$$|\Delta(\omega^2) - 2 \cos \pi\omega| \leq \frac{2\pi M}{|\omega|} e^{\pi(M/|\omega| + |\mu|)}. \tag{3.22}$$

The following theorem has now been established.

Theorem V. $\Delta(\omega^2) = 2 \cos \pi\omega + O(|\omega|^{-1} e^{\pi|\mu|})$ for all large $|\omega|$. Actually, the estimate of Theorem IV holds for all ω since $\Delta(\omega^2)$ is uniformly bounded on the whole real ω-axis[1]

The function $\{\Delta(\omega^2)-\Delta(0)\}/\omega^2$ is clearly entire and $O(|\omega|^{-2} e^{\pi|\mu|})$ thus, the conditions of Theorem I are satisfied with h = 1 and hence one has

Theorem VI. $\Delta(\omega^2) = \Delta(0) + \omega^2 \sum_{j=-\infty}^{\infty} \frac{\Delta(j^2)-\Delta(0)}{j^2} \frac{\sin \pi(\omega-j)}{\pi(\omega-j)}$.

The series converges uniformly in every bounded closed domain of the ω-plane. Similarly, by Theorem V, the function $\Delta(\omega^2) - 2 \cos \pi\omega$ satisfies the conditions of Theorem I. Hence, one has the following theorem.

Theorem VII. $\Delta(\omega^2) = 2\cos \pi\omega + \sum_{j=-\infty}^{\infty} [\Delta(j^2)-2(-1)^j] \frac{\sin \pi(\omega-j)}{\pi(\omega-j)}$.

The series converges uniformly in every bounded closed domain of the ω-plane.

Theorems VI and VII provide representations for the discriminant in terms of its values at the integers. It is now possible to obtain Fourier integral representations of $\Delta(\omega^2)$. One has

Theorem VIII. $\Delta(\omega^2) = \Delta(0) + \omega^2 \int_{-\pi}^{\pi} e^{iu\omega} F(u)\, du$,

$$F(u) = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \frac{\Delta(j^2)-\Delta(0)}{j^2} e^{-iuj} .$$

Proof. The uniform convergence of the Fourier series for $F(u)$ permits termwise integration. The theorem now follows on substitution into the integral formula for $\Delta(\omega^2)$ and reference to Theorem VI.

The function $\{\Delta(\omega^2)-\Delta(0)\}/\omega^2 \in L^2(-\infty,\infty)$, and by Theorem V, the function $\Delta(\omega^2) - 2\cos \pi\omega$ also belongs to $L^2(-\infty,\infty)$. Hence, they are in the closure of the set of functions

$$\left\{ \frac{\sin \pi(\omega-j)}{\pi(\omega-j)} \right\}_{-\infty}^{\infty} .$$

The Parseval relation may now be applied to obtain the following theorems.

Theorem IX. $\int_{-\infty}^{\infty} \left\{ \frac{\Delta(\omega^2)-\Delta(0)}{\omega^2} \right\}^2 d\omega = \sum_{j=-\infty}^{\infty} \left\{ \frac{\Delta(\omega^2)-\Delta(0)}{j^2} \right\}^2$,

Theorem X. $\int_{-\infty}^{\infty} \{\Delta(\omega^2) - 2\cos \pi\omega\}^2 d\omega = \sum_{j=-\infty}^{\infty} \{\Delta(j^2)-2(-1)^j\}^2$.

Theorem III (Paley-Wiener) may be applied to the function $\Delta(\omega^2) - 2\cos \pi\omega$. One obtains

Theorem XI. $\Delta(\omega^2) = 2\cos \pi\omega + \int_{-\pi}^{\pi} e^{i\omega u} G(u)\, du$,

$$G(u) = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \{\Delta(j^2) - 2(-1)^j\} e^{-iuj}.$$

A result of Poisson type may be derived from Theorem VI, namely,

Theorem XII. $\displaystyle\int_{-\infty}^{\infty} \frac{\Delta(\omega^2)-\Delta(0)}{\omega^2}\, d\omega = 2\sum_{j=1}^{\infty} \frac{\Delta(j^2)}{j^2} + \tfrac{1}{2}\Delta''(0) - \frac{\pi^2}{3}\Delta(0).$

Proof. The uniform convergence of the cardinal series in Theorem VI directly yields $(a > 0)$

$$\int_{-a}^{a} \frac{\Delta(\omega^2)-\Delta(0)}{\omega^2}\, d\omega = \sum_{j=-\infty}^{\infty} \frac{\Delta(j^2)-\Delta(0)}{j^2} \int_{-a}^{a} \frac{\sin \pi(\omega-j)}{\pi(\omega-j)}\, d\omega. \tag{3.23}$$

Consider

$$R = \sum_{j=-\infty}^{\infty} \frac{\Delta(j^2)-\Delta(0)}{j^2} - \sum_{j=-\infty}^{\infty} \frac{\Delta(j^2)-\Delta(0)}{j^2} \int_{-a}^{a} \frac{\sin \pi(\omega-j)}{\pi(\omega-j)}\, d\omega. \tag{3.24}$$

Since

$\displaystyle\int_{-\infty}^{\infty} \frac{\sin \pi(\omega-j)}{\pi(\omega-j)}\, d\omega = 1$, one can write R as follows,

$$R = \sum_{j=-\infty}^{\infty} \frac{\Delta(j^2)-\Delta(0)}{j^2} \left[\int_{-\infty}^{-a} \frac{\sin \pi(\omega-j)}{\pi(\omega-j)}\, d\omega + \int_{a}^{\infty} \frac{\sin \pi(\omega-j)}{\pi(\omega-j)}\, d\omega \right]. \tag{3.25}$$

Integration by parts shows that

$$\left| \int_{-\infty}^{-a} \frac{\sin \pi(\omega-j)}{\pi(\omega-j)}\, d\omega + \int_{a}^{\infty} \frac{\sin \pi(\omega-j)}{\pi(\omega-j)}\, d\omega \right| \le \frac{C}{|a-j|}, \tag{3.26}$$

hence,

$$|R| < \frac{C_1}{a} + C_2 \sum_{j=1}^{\infty} \frac{1}{|a-j|\,j^2}. \tag{3.27}$$

The quantities C, C_1, C_2 are constants uniform in $\underline{a}$ and $\underline{j}$.

Let $a = n + \frac{1}{2}$ (n integral), then

$$\sum_{j=1}^{\infty} \frac{1}{|a-j|j^2} = \sum_{1\leq j\leq n} \frac{1}{(a-j)j^2} + \sum_{j>n} \frac{1}{(j-a)j^2} . \qquad (3.28)$$

Since

$$\frac{1}{(a-j)j^2} = \frac{1}{aj^2} + \frac{1}{a^2 j} + \frac{1}{a^2(a-j)} , \qquad (3.29)$$

it follows that

$$\sum_{1 \leq j \leq n} \frac{1}{(a-j)j^2} = O(1/a). \qquad (3.30)$$

Also

$$\frac{1}{(j-a)j^2} = \frac{1}{a} \cdot \frac{1}{j(j-a)} - \frac{1}{aj^2} , \qquad (3.31)$$

and hence

$$\sum_{j > n} \frac{1}{(j-a)j^2} = O(\frac{\ln a}{a^2}). \qquad (3.32)$$

Thus

$$R = O(1/a). \qquad (3.33)$$

Equation (3.33) establishes

$$\int_{-\infty}^{\infty} \frac{\Delta(\omega^2)-\Delta(0)}{\omega^2}\, d\omega = \sum_{j=-\infty}^{\infty} \frac{\Delta(j^2)-\Delta(0)}{j^2} . \qquad (3.34)$$

Since $\Delta(\omega^2)$ is an even function of ω, the power series for $\Delta(\omega^2)$ about the origin has the form

$$\Delta(\omega^2) = \Delta(0) + \omega^2 \frac{\Delta''(0)}{2} + \ldots \qquad (3.35)$$

Hence, the term $j = 0$ in Equation (3.34) is $\frac{1}{2}\Delta''(0)$. Use of the well-known expansion

$$\sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6} \qquad (3.36)$$

now yields Theorem XII.

Theorem II yields representations for $\Delta(\omega^2)$ in terms of the values of Δ and Δ' at the integers. The theorem may be applied as above to the functions $\{\Delta(\omega^2)-\Delta(0)\}/\omega^2$ and $\Delta(\omega^2)-2\cos\pi\omega$ with $h = 2$. The results are embodied in Theorems XIII and XIV below.

Theorem XIII.

$$\Delta(\omega^2) = \Delta(0) + \omega^2 \sum_{j=-\infty}^{\infty} \left[\frac{\sin \pi/2\,(\omega-2j)}{\pi/2\,(\omega-2j)}\right]^2 \left[(\frac{\omega}{j}-1)\,\frac{\Delta(4j^2)-\Delta(0)}{4j^2} + (\frac{\omega}{j}-2)\Delta'(4j^2)\right].$$

Theorem XIV.

$$\Delta(\omega^2) = 2\cos\pi\omega + \sum_{j=-\infty}^{\infty} \left[\frac{\sin \pi/2\,(\omega-2j)}{\pi/2\,(\omega-2j)}\right]^2 [\Delta(4j^2)-2+(\omega-2j)2j\Delta'(4j^2)].$$

In Theorems XIII and XIV the series converge uniformly in every bounded closed domain of the ω-plane.

IV. A Transformation of Hill's Equation and an Asymptotic Expansion for the Discriminant

Consider the substitution

$$y(x) = e^{i\int w dx} \tag{4.1}$$

in Equation (1.1). Since

$$y'(x) = iwy, \tag{4.2}$$

$$y''(x) = iwy - w^2y \tag{4.3}$$

one obtains the Riccati equation

$$iw' - w^2 + \omega^2 + g = 0. \tag{4.4}$$

A perturbation form of solution of WKB type[8] will be obtained by the introduction of a small perturbation parameter ε as follows

$$\varepsilon iw' - w^2 + \omega^2 + g = 0. \tag{4.5}$$

Equation (4.4) is obtained from Equation (4.5) on setting $\varepsilon = 1$. Let

$$w = w_o + \varepsilon w_1 + \dots , \tag{4.6}$$

then

$$w_o = (\omega^2 + g)^{\frac{1}{2}} , \tag{4.7}$$

$$w_1 = \tfrac{1}{4} i \frac{g'}{\omega^2+g} .$$

Thus

$$w = (\omega^2 + g)^{\frac{1}{2}} + \tfrac{1}{4}\varepsilon i \frac{g'}{\omega^2+g} + \dots \quad . \tag{4.8}$$

Setting $\varepsilon = 1$, and substituting in Equation (4.1) yields

$$y(x) \cong (\omega^2 + g)^{-\frac{1}{4}} e^{i\int_0^x (\omega^2+g)^{\frac{1}{2}} d\xi} \tag{4.9}$$

The above procedure renders plausible the approximation of Equation (4.9). In order to obtain exact results, another method is required leading to the same approximate solution but permitting the estimation of error. For that purpose, the following change of variables is suggested by the form of Equation (4.9). Let

$$h = (\omega^2 + g)^{\frac{1}{4}}\, y, \tag{4.10}$$

$$\eta = \int_0^x (\omega^2 + g)^{\frac{1}{2}}\, d\xi, \tag{4.11}$$

in which ω is so large that $\omega^2 + g \geq 0$. Considering h as a function of η, one has

$$\frac{dh}{d\eta} = \tfrac{1}{4}(\omega^2 + g)^{-5/4}\, g'y + (\omega^2 + g)^{-\frac{1}{4}}\, y, \tag{4.12}$$

$$\frac{d^2h}{d\eta^2} = -\frac{5}{16}\, (\omega^2+g)^{-\frac{1}{4}}\, g'^2 y + \tfrac{1}{4}(\omega^2+g)^{-7/4}\, g''y + (\omega^2+g)^{-3/4}\, y''. \tag{4.13}$$

Thus Equation (1.1) is transformed to

$$\frac{d^2h}{d\eta^2} + h + Fh = 0, \tag{4.14}$$

$$F = -\frac{1}{4}\, \frac{g''}{(\omega^2+g)^2} + \frac{5}{16}\, \frac{g'^2}{(\omega^2+g)^3}\ . \tag{4.15}$$

For the purpose of approximation, it is best to rewrite Equation (4.14) in integral equation form. The required integral equation is

$$h(\eta) = \ell(\eta) - \int_0^\eta \sin\,(\eta-u)K(u)h(u)\, du, \tag{4.16}$$

$$\ell(\eta) = h(0)\, \cos\, \eta + \left.\frac{dh}{d\eta}\right]_{\eta=0} \sin\, \eta\ ,$$

$$K(\eta) = F(x).$$

For the determination of $y_1(x)$, the initial conditions on $h(\eta)$ are

$$h(0) = (\omega^2 + \alpha)^{\frac{1}{4}}, \tag{4.17}$$

$$\left.\frac{dh}{d\eta}\right]_{\eta=0} = \tfrac{1}{4}\, \beta(\omega^2 + \alpha)^{-5/4},$$

in which

$$g(0) = \alpha, \qquad (4.18)$$

$$g'(0) = \beta.$$

The function $h_1(\eta)$ is defined by the initial conditions of Equation (4.17). One has from Equation (4.16)

$$h_1(\eta) = \ell_1(\eta) - \int_0^\eta \sin(\eta-u)K(u)h_1(u)\,du, \qquad (4.19)$$

$$\ell_1(\eta) = (\omega^2 + \alpha)^{\frac{1}{4}} \cos\eta + \tfrac{1}{4}\beta(\omega^2 + \alpha)^{-5/4} \sin\eta.$$

Similarly for the determination of $y_2(x)$, the function $h_2(\eta)$ is defined with the initial conditions

$$h_2(0) = 0, \qquad (4.20)$$

$$\left.\frac{dh_2}{d\eta}\right]_{\eta=0} = (\omega^2 + \alpha)^{-\frac{1}{4}}.$$

The function $h_2(\eta)$ satisfies

$$h_2(\eta) = \ell_2(\eta) - \int_0^\eta \sin(\eta-u)K(u)h_2(u)\,du, \qquad (4.21)$$

$$\ell_2(\eta) = (\omega^2 + \alpha)^{-\frac{1}{4}} \sin\eta.$$

Equations (4.19) and (4.21) may be solved by successive approximations. Define the sequences $\{L_k\}_0^\infty$, $\{M_k\}_0^\infty$ by

$$L_k(\eta) = -\int_0^\eta \sin(\eta-u)K(u)L_{k-1}(u)\,du, \qquad (4.22)$$

$$L_0(\eta) = \ell_1(\eta),$$

and

$$M_k(\eta) = -\int_0^\eta \sin(\eta-u)K(u)M_{k-1}(u)\,du, \qquad (4.23)$$

$$M_0(\eta) = \ell_2(\eta),$$

then

$$h_1(\eta) = \sum_{k=0}^{\infty} L_k(\eta), \tag{4.24}$$

$$h_2(\eta) = \sum_{k=0}^{\infty} M_k(\eta). \tag{4.25}$$

From Equation (4.15) one has

$$F(x) = K(\eta) = O(\omega^{-4}) \tag{4.26}$$

uniformly in x. Since, from Equation (4.19)

$$L_0 = \ell_1 = O(\omega^{\frac{1}{2}}) \tag{4.27}$$

uniformly in η, one obtains, by induction on Equation (4.22),

$$L_k = O(\omega^{-4k+\frac{1}{2}} \frac{\eta^k}{k!}) . \tag{4.28}$$

Also since from Equation (4.21)

$$M_0 = \ell_2 = O(\omega^{-\frac{1}{2}}), \tag{4.29}$$

one obtains, by induction on Equation (4.23),

$$M_k = O(\omega^{-4k-\frac{1}{2}} \frac{\eta^k}{k!}) . \tag{4.30}$$

From Equation (4.10) one obtains

$$y'(x) = -\tfrac{1}{4}g'(\omega^2 + g)^{-5/4} h + (\omega^2 + g)^{\frac{1}{4}} \frac{dh}{d\eta} . \tag{4.31}$$

Thus in order to determine the discriminant, it will be necessary to evaluate $dh_2/d\eta$. Equation (4.24) gives

$$\frac{dh_2}{d\eta} = \sum_{k=0}^{\infty} \frac{dM_k}{d\eta} . \tag{4.32}$$

From Equation (4.23) one obtains

$$\frac{dM_k}{d\eta} = - \int_0^{\eta} \cos(\eta-u)K(u)M_{k-1}(u)\, du. \tag{4.33}$$

It follows that $dM_k/d\eta$ obeys the same order relation as M_k itself, that is,

$$\frac{dM_k}{d\eta} = O(\omega^{-4k-\frac{1}{2}} \frac{\eta^k}{k!}) \ . \tag{4.34}$$

The function $y_1(\pi)$ is given by

$$y_1(\pi) = (\omega^2 + \alpha)^{-\frac{1}{4}} \sum_{k=0}^{\infty} L_k(\bar{\bar{\eta}}) \ , \tag{4.35}$$

$$\bar{\bar{\eta}} = \int_0^{\pi} (\omega^2 + g)^{\frac{1}{2}} \, d\xi \ .$$

Also the function $y_2'(\pi)$ is given by

$$y_2'(\pi) = -\tfrac{1}{4}\beta(\omega^2+\alpha)^{-5/4} \sum_{k=0}^{\infty} M_k(\bar{\bar{\eta}}) + (\omega^2+\alpha)^{\frac{1}{4}} \sum_{k=0}^{\infty} \frac{dM_k(\bar{\bar{\eta}})}{d\eta} \ . \tag{4.36}$$

Define $\delta_k(\omega^2)$ by

$$\delta_k(\omega^2) = (\omega^2+\alpha)^{-\frac{1}{4}} L_k(\bar{\bar{\eta}}) - \tfrac{1}{4}\beta(\omega^2+\alpha)^{-5/4} M_k(\bar{\bar{\eta}}) + (\omega^2+\alpha)^{\frac{1}{4}} \frac{dM_k(\bar{\bar{\eta}})}{d\eta}, \tag{4.37}$$

then the discriminant $\Delta(\omega^2)$ is given by

$$\Delta(\omega^2) = \sum_{k=0}^{\infty} \delta_k(\omega^2). \tag{4.38}$$

Equations (4.28), (4.30), and (4.34) show that

$$\delta_k(\omega^2) = O(\omega^{-4k} \frac{\eta^k}{k!}). \tag{4.39}$$

From Equation (4.35) one has

$$\bar{\bar{\eta}} = O(\omega), \tag{4.40}$$

hence,

$$\delta_k(\omega^2) = O(\omega^{-3k} /k!) \ . \tag{4.41}$$

The sum $\sum_{k=n+1}^{\infty} \delta_k(\omega^2)$ may now be readily estimated. Thus

$$\sum_{k=n+1}^{\infty} \delta_k(\omega^2) = O\left(\sum_{k=n+1}^{\infty} \frac{\omega^{-3k}}{k!} \right) = O(\omega^{-3n-3}). \tag{4.42}$$

One may now state the following theorem.

Theorem XV. $\Delta(\omega^2) = \sum_{k=0}^{n} \delta_k(\omega^2) + O(|\omega|^{-3n-3})$ for all real $\omega \neq 0$. The first three δ_k are given by

$$\delta_0 = 2 \cos \bar{\eta}, \tag{4.43}$$

$$\delta_1 = -\sin \bar{\eta} \int_0^{\bar{\eta}} K(u)\, du, \tag{4.44}$$

$$\delta_2 = \int_0^{\bar{\eta}} \int_0^{u} \sin (\bar{\eta}-u+u_1) \sin (u-u_1)\, K(u)\, K(u_1)\, du_1\, du. \tag{4.45}$$

Since

$$u = \int_0^x (\omega^2 + g)^{\frac{1}{2}}\, d\xi, \tag{4.46}$$

one has

$$u = \omega x + \tfrac{1}{2}\omega^{-1} \int_0^x g\, d\xi - \tfrac{1}{8}\omega^{-3} \int_0^x g^2 d\xi \tag{4.47}$$

$$+ \frac{1}{16}\omega^{-5} \int_0^x g^3 d\xi - \frac{5}{128}\omega^{-7} \int_0^x g^4 d\xi + O(|\omega|^{-9}),$$

and, hence, $\bar{\eta}$ is given by

$$\bar{\eta} = \pi\omega - \frac{1}{8}\omega^{-3} \int_0^{\pi} g^2 d\xi + \frac{1}{16}\omega^{-5} \int_0^{\pi} g^3 d\xi - \frac{1}{128}\omega^{-7} \int_0^{\pi} g^4 d\xi \tag{4.48}$$

$$+ O(|\omega|^{-9}).$$

One now obtains for δ_0

$$\delta_0 = A_0 \cos \pi\omega + B_0 \sin \pi\omega + O(|\omega|^{-9}), \tag{4.49}$$

$$A_0 = 2 - \frac{1}{64}\left(\int_0^\pi g^2 d\xi\right)^2 \omega^{-6} + \frac{1}{64}\int_0^\pi g^3\, d\xi \cdot \omega^{-8},$$

$$B_0 = \frac{1}{4}\int_0^\pi g^2 d\xi \cdot \omega^{-3} - \frac{1}{8}\int_0^\pi g^3\, d\xi \cdot \omega^{-5} + \frac{5}{64}\int_0^\pi g^4 d\xi \cdot \omega^{-7}.$$

For the computation of δ_1, the change of variables given by Equation (4.46) is used in the integral. Thus,

$$\int_0^{\bar{\bar{\eta}}} K(u)du = \int_0^\pi F(x)\ (\omega^2 + g)^{\frac{1}{2}}\ dx. \tag{4.50}$$

From Equation (4.15) one has

$$F(x) = -\tfrac{1}{4}g''\omega^{-4} + (\tfrac{1}{2}gg'' + \frac{5}{16}g'^2)\omega^{-6} - (\frac{3}{4}\,g^2g'' + \frac{5}{16}\,gg'^2)\omega^{-8} + O(\ \omega^{-10}) \tag{4.51}$$

and since

$$(\omega^2+g)^{\frac{1}{2}} = \omega + \tfrac{1}{2}\omega^{-1}g - \frac{1}{8}\omega^{-3}g^2 + \frac{1}{16}\omega^{-5}g^3 - \frac{5}{128}\omega^{-7}g^4 + O(|\omega|^{-9}), \tag{4.52}$$

one obtains

$$F(x)(\omega^2+g)^{\frac{1}{2}} = -\tfrac{1}{4}g''\omega^{-3} + (\frac{3}{8}gg''+\frac{5}{16}g'^2)\omega^{-5} - (\frac{15}{32}g^2g'' + \frac{25}{32}gg'^2)\omega^{-7} + O(|\omega|^{-9}). \tag{4.53}$$

Thus

$$\int_0^\pi F(x)(\omega^2+g)^{\frac{1}{2}}dx = -\frac{1}{16}\int_0^\pi g'^2 d\xi\cdot\omega^{-5} + \frac{5}{32}\int_0^\pi gg'^2 d\xi\cdot\omega^{-7} + O(|\omega|^{-9}) \tag{4.54}$$

Also

$$\sin\bar{\bar{\eta}} = \sin\pi\omega - \frac{\omega^{-3}}{8}\int_0^\pi g^2 d\xi \cdot \cos\pi\omega + O(|\omega|^{-5}) \tag{4.55}$$

and hence

$$\delta_1 = A_1\cos\pi\omega + B_1\sin\pi\omega + O(|\omega|^{-9}), \tag{4.56}$$

$$A_1 = -\frac{1}{128}\int_0^\pi g^2 d\xi\int_0^\pi g'^2\ d\xi \cdot \omega^{-8},$$

$$B_1 = \frac{1}{16}\int_0^\pi g'^2 d\xi\cdot\omega^{-5} - \frac{5}{32}\int_0^\pi gg'^2 d\xi\cdot\omega^{-7}.$$

The determination of the asymptotic expansion for δ_2 is somewhat more involved. After using the change of variables of Equation (4.46), Equation (4.45) becomes

$$\delta_2 = \int_0^\pi \int_0^x \sin(\bar{\bar{\eta}}-u+u_1)\sin(u-u_1)F(x)(\omega^2+g(x))^{\frac{1}{2}}F(x_1)\cdot (\omega^2+g(x_1))^{\frac{1}{2}}dx_1dx. \tag{4.57}$$

Using a trigonometric identity, Equation (4.57) may be written as follows:

$$\delta_2 = \tfrac{1}{2}\int_0^\pi\int_0^x \cos(\bar{\bar{\eta}}-2u+2u_1)F(x)(\omega^2+g(x))^{\frac{1}{2}}F(x_1)(\omega^2+g(x_1))^{\frac{1}{2}}dx_1dx \tag{4.58}$$

$$-\tfrac{1}{2}\cos\bar{\bar{\eta}}\int_0^\pi\int_0^x F(x)(\omega^2+g(x))^{\frac{1}{2}}F(x_1)(\omega^2+g(x_1))^{\frac{1}{2}}dx_1dx.$$

Integration by parts and Equation (4.54) establish that

$$\int_0^\pi\int_0^x F(x)(\omega^2+g(x))^{\frac{1}{2}}F(x_1)(\omega^2+g(x_1))^{\frac{1}{2}}dx_1dx = \tfrac{1}{2}\left(\int_0^\pi F(x)(\omega^2+g(x))^{\frac{1}{2}}dx\right)^2 \tag{4.59}$$

$$= O(\omega^{-10}).$$

Thus

$$\delta_2 = \tfrac{1}{2}\int_0^\pi\int_0^x \cos(\bar{\bar{\eta}}-2u+2u_1)F(x)(\omega^2+g(x))^{\frac{1}{2}}F(x_1)(\omega^2+g(x_1))^{\frac{1}{2}}dx_1dx \tag{4.60}$$

$$+O(\omega^{-10}).$$

From Equation (4.47) one has

$$\bar{\bar{\eta}} - 2u + 2u_1 = \omega(\pi-2x+2x_1) - \omega^{-1}\int_{x_1}^x g\,d\xi + O(|\omega|^{-3}). \tag{4.61}$$

Thus,

$$\cos(\bar{\eta}-2u+2u_1) = \left[1-\tfrac{1}{2}\omega^{-2}\left(\int_{x_1}^{x} g\,d\xi\right)^2\right]\cos\,\omega(\pi-2x+2x_1)$$
$$+\,\omega^{-1}\int_{x_1}^{x} g\,d\xi\,\sin\,\omega(\pi-2x+2x_1) + O(|\omega|^{-3}). \tag{4.62}$$

Equations (4.62) and (4.53) yield

$$\cos\,(\bar{\eta}-2u+2u_1)F(x)(\omega^2+g(x))^{\frac{1}{2}}F(x_1)(\omega^2+g(x_1))^{\frac{1}{2}} = \tag{4.63}$$
$$\left[\frac{1}{16}g''(x)g''(x_1)\omega^{-6} - \left\{\left(\frac{3}{32}g(x_1)g''(x_1)g''(x) + \frac{5}{64}g'(x_1)^2g''(x)\right)\right.\right.$$
$$+\left(\frac{3}{32}\,g(x)g''(x)g''(x_1) + \frac{5}{64}\,g'(x)^2g''(x_1)\right)$$
$$\left.\left.+\,\frac{1}{32}\,g''(x)g''(x_1)\left(\int_{x_1}^{x} g\,d\xi\right)^2\right\}\omega^{-8}\right]\;\cos\,\omega(\pi-2x+2x_1)$$
$$+\,\frac{1}{16}\,g''(x)g''(x_1)\omega^{-7}\int_{x_1}^{x} g\,d\xi\cdot\sin\,\omega(\pi-2x+2x_1) + O(|\omega|^{-9}).$$

On the assumption that $g(x)$ has an integrable third derivative, one may apply integration by parts to obtain

$$\int_0^x\int_0^x\left\{\left(\frac{3}{32}\,g(x_1)g''(x_1)g''(x) + \frac{5}{64}\,g'(x_1)^2g''(x)\right)\right. \tag{4.64}$$
$$+\left(\frac{3}{32}\,g(x)g''(x)g''(x_1) + \frac{5}{64}\,g'(x)^2g''(x_1)\right)$$
$$\left.+\,\frac{1}{32}g''(x)g''(x_1)\left(\int_{x_1}^{x} g\,d\xi\right)^2\right\}\cos\,\omega(\pi-2x+2x_1)dx_1dx = O(|\omega|^{-1}).$$

Thus

$$\delta_2 = \frac{1}{32}\,\omega^{-6}\int_0^{\pi}\int_0^{x} g''(x)g''(x_1)\cos\omega(\pi-2x+2x_1)dx_1\,dx \tag{4.65}$$

$$+\frac{1}{32}\,\omega^{-7}\int_0^{\pi}\int_0^{x} g''(x)g''(x_1)\left(\int_{x_1}^{x} g\,d\xi\right)\sin\omega(\pi-2x+2x_1)dx_1\,dx$$

$$+\ O(|\omega|^{-9}).$$

Since

$$\int_0^{x} g''(x_1)\cos\omega(\pi-2x+2x_1)dx_1 = \tfrac{1}{2}\omega^{-1}g''(x)\sin\pi\omega \tag{4.66}$$

$$-\tfrac{1}{2}\omega^{-1}g''(o)\sin\omega(\pi-2x) - \tfrac{1}{2}\omega^{-1}\int_0^{x} g'''(x_1)\sin\omega(\pi-2x+2x_1)\,dx_1\ ,$$

one has

$$\delta_2 = \frac{1}{64}\omega^{-7}\sin\pi\omega\int_0^{\pi} g''(x)^2dx - \frac{1}{64}\omega^{-7}g''(o)\int_0^{\pi} g''(x)\sin\omega(\pi-2x)dx \tag{4.67}$$

$$-\frac{1}{64}\omega^{-7}\int_0^{\pi} g''(x)dx\int_0^{x} g'''(x_1)\sin\omega(\pi-2x+2x_1)dx_1$$

$$+\frac{1}{32}\omega^{-7}\int_0^{\pi}\int_0^{x} g'''(x)g''(x_1)\left(\int_{x_1}^{x} g\,d\xi\right)\sin\omega(\pi-2x+2x_1)dx_1dx + O(|\omega|^{-9}).$$

Integration by parts yields

$$\int_0^{\pi} g''(x)\sin\omega(\pi-2x)dx = \tfrac{1}{2}\omega^{-1}\int_0^{\pi} g'''(x)\cos\omega(\pi-2x)dx. \tag{4.68}$$

One now assumes, additionally, that $g'''(x)$ is of bounded variation. Then, by a well-known lemma (this follows directly from the representation of a function of bounded variation as the difference of two monotone functions, and the second mean value theorem),

$$\int_0^{\pi} g'''(x)\cos\omega(\pi-2x)dx = O(|\omega|^{-1}) \tag{4.69}$$

and, hence,

$$-\frac{1}{64}\,\omega^{-7}g''(o)\int_o^{\pi} g''(x)\sin\,\omega(\pi-2x)dx = O(|\omega|^{-9}). \tag{4.70}$$

In the first iterated integral of Equation (4.67), it is advantageous to interchange the order of integration. Thus

$$\int_o^{\pi} g''(x)dx \int_o^{x} g'''(x_1)\sin\,\omega(\pi-2x+2x_1)dx_1 = \tag{4.71}$$

$$\int_o^{\pi} g'''(x_1)dx_1 \int_{x_1}^{\pi} g''(x)\sin\,\omega(\pi-2x+2x_1)dx.$$

Integration by parts yields

$$\int_{x_1}^{\pi} g''(x)\sin\,\omega(\pi-2x+2x_1)dx = \tfrac{1}{2}\omega^{-1}g''(o)\cos\,\omega(\pi-2x_1) \tag{4.72}$$

$$-\tfrac{1}{2}\omega^{-1}g''(x_1)\,\cos\,\pi\omega$$

$$-\tfrac{1}{2}\omega^{-1}\int_{x_1}^{\pi} g'''(x)\,\cos\,\omega(\pi-2x+2x_1)\,dx.$$

Thus

$$-\frac{1}{64}\omega^{-7}\int_o^{\pi} g''(x)dx\int_o^{x} g'''(x_1)\sin\,\omega(\pi-2x+2x_1)dx_1 = \tag{4.73}$$

$$-\frac{1}{128}\omega^{-8}g''(o)\int_o^{\pi} g'''(x_1)\cos\,\omega(\pi-2x_1)dx_1$$

$$-\frac{1}{128}\omega^{-8}\cos\,\pi\omega\int_o^{\pi} g'''(x_1)g''(x_1)dx$$

$$+\frac{1}{128}\omega^{-8}\int_o^{\pi} g'''(x_1)dx_1\int_{x_1}^{\pi} g'''(x)\cos\,\omega(\pi-2x+2x_1)dx.$$

Since $g'''(x)$ is of bounded variation, one has

$$\int_0^{\pi} g''' (x_1) \cos \omega(\pi-2x_1)dx_1 = O(|\omega|^{-1}), \tag{4.74}$$

$$\int_{x_1}^{\pi} g''' (x) \cos \omega(\pi-2x+2x_1)dx = O(|\omega|^{-1}),$$

also

$$\int_0^{\pi} g''' (x_1)g''(x_1)dx_1 = 0. \tag{4.75}$$

Hence, the entire integral expression on the left hand side of Equation (4.73) is $O(|\omega|^{-9})$. One now has

$$\delta_2 = \frac{1}{64}\omega^{-7}\sin \pi\omega \int_0^{\pi} g''(x)^2dx \tag{4.76}$$

$$+ \frac{1}{32}\omega^{-7} \int_0^{\pi} g''(x)dx \int_0^{x} g''(x_1)\left(\int_{x_1}^{x} gd\xi\right)\sin \omega(\pi-2x+2x_1)dx_1 + O(|\omega|^{-9}).$$

Applying integration by parts, one has

$$\int_0^{x} g''(x_1)\left(\int_{x_1}^{x} gd\xi\right)\sin \omega(\pi-2x+2x_1)dx_1 = \tfrac{1}{2}\omega^{-1}g''(o)\cos \omega(\pi-2x)\cdot\int_0^{x} gd\xi \tag{4.77}$$

$$+ \tfrac{1}{2}\omega^{-1} \int_0^{x}\left\{g''' (x_1)\left(\int_{x_1}^{x} gd\xi\right) - g''(x_1)g(x_1)\right\} \cos \omega(\pi-2x+2x_1)dx_1 .$$

From the bounded variation of

$$g''' (x_1) \left(\int_{x_1}^{x} gd\xi\right) - g''(x_1)g(x_1)$$

one has

$$\int_0^{x}\left\{g''' (x_1)\left(\int_{x_1}^{x} gd\xi\right) - g''(x_1)g(x_1)\right\} \cos \omega(\pi-2x+2x_1)dx_1 \tag{4.78}$$

$$= O(\omega^{-1})$$

Hence

$$\delta_2 = \frac{1}{64}\omega^{-7}\sin\pi\omega\int_0^\pi g''(x)^2dx \tag{4.79}$$

$$+\frac{1}{64}\omega^{-8}g''(0)\int_0^\pi g''(x)\left(\int_0^x gd\xi\right)\cos\omega(\pi-2x)dx + O(|\omega|^{-9}).$$

Since

$$g''(x)\left(\int_0^x gd\xi\right)$$

is of bounded variation, one finally concludes that

$$\delta_2 = A_2\cos\pi\omega + B_2\sin\pi\omega + O(|\omega|^{-9}), \tag{4.80}$$

$$A_2 = 0,$$

$$B_2 = \frac{1}{64}\int_0^\pi g''^2d\xi\omega^{-7}$$

Theorem XV and the above determination of δ_0, δ_1, δ_2, now yield the following asymptotic expression for the discriminant.

Theorem XVI. $g'''(x)$ is of bounded variation over $(0, \pi)$

$$\Rightarrow \Delta(\omega^2) = A\cos\pi\omega + B\sin\pi\omega + O(|\omega|^{-9}),$$

$$A = 2 - \frac{1}{64}\left(\int_0^\pi g^2d\xi\right)^2\omega^{-6} + \left[\frac{1}{64}\int_0^\pi g^2d\xi\int_0^\pi g^3d\xi - \frac{1}{128}\int_0^\pi g^2d\xi\int_0^\pi g'^2d\xi\right]\omega^{-8},$$

$$B = \frac{1}{4}\int_0^\pi g^2d\xi\cdot\omega^{-3} + \left[\frac{1}{16}\int_0^\pi g'^2d\xi - \frac{1}{8}\int_0^\pi g^3d\xi\right]\omega^{-5}$$

$$+\left[\frac{5}{64}\int_0^\pi g^4d\xi - \frac{5}{32}\int_0^\pi gg'^2d\xi + \frac{1}{64}\int_0^\pi g''^2d\xi\right]\omega^{-7}.$$

The form of $\Delta(\omega^2)$ in Theorem XVI may be greatly simplified on observing that $\sqrt{A^2+B^2} = 2 + O(|\omega|^{-9})$. Thus one has

$$\Delta(\omega^2) = 2 \cos \pi\phi + O(|\omega|^{-9}) \; , \tag{4.81}$$

$$\phi = \omega - \frac{1}{8\pi} \int_0^\pi g^2 d\xi \cdot \omega^{-3} + \left[\frac{1}{16\pi} \int_0^\pi g^3 d\xi - \frac{1}{32\pi} \int_0^\pi g'^2 d\xi \right] \omega^{-5}$$

$$- \left[\frac{5}{128\pi} \int_0^\pi g^4 d\xi - \frac{5}{64\pi} \int_0^\pi g g'^2 d\xi + \frac{1}{128\pi} \int_0^\pi g''^2 d\xi \right] \omega^{-7} \; .$$

It may be remarked that the coefficients of the powers of ω in Equation (4.81) are final, that is, extending the asymptotic expansion beyond $O(|\omega|^{-9})$ does not change the coefficients already found.

V. Asymptotic Expansions for the Eigenvalues

It is the present purpose to determine expansions asymptotic for large n of the eigenvalues λ_{2n-1}, λ_{2n} for which Equation (1.1) has solutions of period π and λ'_{2n-1}, λ'_{2n} for which Equation (1.1) has solutions of period 2π. To this end, the explicit asymptotic expansion of the discriminant given in Equation (4.81) will be employed. However, it will be necessary to establish an additional structure theorem for the discriminant.

It is known[1], or can be shown directly from Theorem XV, that for each $\nu \geq 0$ one has

$$\Delta(\omega^2) = A \cos \pi\omega + B \sin \pi\omega + O(\omega^{-2\nu-4}) \tag{5.1}$$

in which there are constants a_{2j}, b_{2j-1} so that

$$A = 2 + \sum_{j=3}^{\nu+1} a_{2j}\, \omega^{-2j}, \tag{5.2}$$

$$B = \sum_{j=2}^{\nu+2} b_{2j-1}\, \omega^{-2j+1}. \tag{5.3}$$

It follows that $\Delta(\omega^2)$ may be put into the form

$$\Delta(\omega^2) = \sqrt{A^2 + B^2}\, \cos \pi\varphi + O(\omega^{-2\nu-4}) \tag{5.4}$$

in which there are constants C_{2j-1} so that

$$\varphi = \omega + \sum_{j=2}^{\nu+2} C_{2j-1}\, \omega^{-2j+1}. \tag{5.5}$$

The constants C_{2j-1} are, of course, functionals of $g(x)$ and are clearly independent of ν.

Corresponding to the eigenvalues λ_{2n-1}, λ_{2n}, λ'_{2n-1}, λ'_{2n} are the quantities ω_{2n-1}, ω_{2n}, ω'_{2n-1}, ω'_{2n}, respectively, in which $\lambda = \omega^2$. It is known that[1] ω_{2n-1}, ω_{2n} are the zeros of $\Delta(\omega^2) - 2$, and ω'_{2n-1}, ω'_{2n} are the zeros of $\Delta(\omega^2) + 2$. Thus the equations

$$\sqrt{A^2 + B^2}\, \cos \pi\varphi = 2 + O(\omega^{-2\nu-4}), \tag{5.6}$$

$$\sqrt{A^2 + B^2}\, \cos \pi\varphi = -2 + O(\omega^{-2\nu-4} \tag{5.7}$$

serve to determine asymptotic formulae for ω_{2n-1}, ω_{2n}, ω'_{2n-1}, ω'_{2n}, respectively. A theorem of Hochstadt[7] states that if $g(x)$ has derivatives up to order ν, then

$$\omega_{2n}-\omega_{2n-1} = O(n^{-\nu-2}), \tag{5.8}$$

$$\omega'_{2n}-\omega'_{2n-1} = O(n^{-\nu-2}). \tag{5.9}$$

The following theorem may now be established.

Theorem XVII. Let $g^{(\nu)}(x)$ exist and be integrable for $\nu \geq 1$, then there exist constants C_{2j-1}, independent of ν so that

$$\Delta(\omega^2) = 2 \cos \pi\varphi + O(\omega^{-2\nu-4}),$$

$$\varphi = \omega + \sum_{j=2}^{\nu+2} C_{2j-1}\, \omega^{-2j+1} .$$

Proof. In view of Equations (5.4) and (5.5), it is necessary to prove only that

$$\sqrt{A^2 + B^2} = 2 + O(\omega^{-2\nu-4}). \tag{5.10}$$

Consider Equation (5.6). Let $\omega > \Omega > 0$ in which Ω is a constant, and let $K > 0$ be a constant, then the inequality

$$\sqrt{A^2 + B^2} < 2 - K\omega^{-2\nu-4} , \quad \omega > \Omega \tag{5.11}$$

is contradictory to the known existence of the infinitely many eigenvalues ω_{2n-1}, ω_{2n} since

$$2 + O(\omega^{-2\nu-4}) - \sqrt{A^2 + B^2} \cos \pi\varphi \geq 2 + O(\omega^{-2\nu-4}) - \sqrt{A^2 + B^2} > 0. \tag{5.12}$$

The inequality

$$\sqrt{A^2 + B^2} > 2 + K\omega^{-2\nu-4}, \quad \omega > \Omega \tag{5.13}$$

implies the existence of two solutions, say φ_1, φ_2, of Equation (5.6) in the neighborhood of every sufficiently large even integer $2n$. Let

$$\sqrt{A^2 + B^2} = 2 + c, \tag{5.14}$$

$$c > K\omega^{-2\nu-4}, \quad \omega > \Omega, \tag{5.15}$$

and let Equation (5.6) be rewritten in the form

$$(2 + c) \cos \pi\varphi = 2 + d, \quad d = O(\omega^{-2\nu-4}) \tag{5.16}$$

also, set $\varphi = 2n + \delta$ where δ is small, then, since $\cos \pi\varphi = \cos \pi\delta$, the two values of δ are given by

$$\delta_1 = E^{\frac{1}{2}} + O(E), \tag{5.17}$$

$$\delta_2 = -E^{\frac{1}{2}} + O(E),$$

$$E = \frac{1}{\pi^2}\,\frac{c-d}{1+\frac{1}{2}c}\,.$$

The above values of δ_1, δ_2 were obtained from the expansion of $\cos \pi\delta$ in the neighborhood of the origin. Let $\varphi_1 = 2n + \delta_1$, $\varphi_2 = 2n + \delta_2$, then $\delta_1 - \delta_2 = \varphi_1 - \varphi_2$ and

$$\delta_1 - \delta_2 = 2E^{\frac{1}{2}} + O(E)\,. \tag{5.18}$$

Equations (5.15), (5.16), and (5.18) show that

$$\delta_1 - \delta_2 > O(\omega^{-\nu-2})\,. \tag{5.19}$$

From Equation (5.5) one has

$$\omega_{2n-1} = 2n + \varepsilon_2, \tag{5.20}$$

$$\omega_{2n} = 2n + \varepsilon_1,$$

in which

$$\delta_1 \sim \varepsilon_1\,, \quad \delta_2 \sim \varepsilon_2. \tag{5.21}$$

Hence, from Equation (5.19)

$$\omega_{2n} - \omega_{2n-1} = \varepsilon_1 - \varepsilon_2 > O(\omega^{-\nu-2}) = O(n^{-\nu-2}). \tag{5.22}$$

Equation (5.22) contradicts Equation (5.8). The argument starting from Equation (5.7) is the same as the above except that $\varphi_1 = 2n - 1 + \delta_1$, $\varphi_2 = 2n - 1 + \delta_2$ and a contradiction is obtained to Equation (5.9). The lack of dependence of the C_{2j-1} on ν follows from the finality of determination of the a_{2j}, b_{2j-1} in Equations (5.2) and (5.3). Since there are infinitely many ω_{2n-1}, $\omega_{2n} > \Omega$ and since $\omega^{2\nu+2}A$, $\omega^{2\nu+3}B$ are polynomials, Equation (5.10) is true for all $\omega \neq 0$. It is now possible to establish the following theorem giving the asymptotic developments of the eigenvalues.

Theorem XVIII. $g^{(6)}(x)$ exists and is integrable

$$\Rightarrow \lambda_{2n-1} = 4n^2 + \frac{D_2}{(4n)^2} + \frac{D_4}{(4n)^4} + \frac{D_6}{(4n)^6} + O(n^{-7}),$$

$$\lambda_{2n} = 4n^2 + \frac{D_2}{(4n)^2} + \frac{D_4}{(4n)^4} + \frac{D_6}{(4n)^6} + O(n^{-7}),$$

$$\lambda'_{2n-1} = (2n-1)^2 + \frac{D_2}{(4n-2)^2} + \frac{D_4}{(4n-2)^4} + \frac{D_6}{(4n-2)^6} + O(n^{-7}),$$

$$\lambda'_{2n} = (2n-1)^2 + \frac{D_2}{(4n-2)^2} + \frac{D_4}{(4n-2)^4} + \frac{D_6}{(4n-2)^6} + O(n^{-7}),$$

in which

$$D_2 = 1/\pi \int_0^\pi g^2 \, dx,$$

$$D_4 = 1/\pi \int_0^\pi \{g'^2 - 2g^3\} \, dx,$$

$$D_6 = 1/\pi \int_0^\pi \{5g^4 - 10gg'^2 + 25g''^2\} \, dx - 5D_2^2.$$

Proof. Theorem XVII permits one to write

$$\Delta(\omega^2) = 2 \cos \pi\varphi + O(\omega^{-16}), \tag{5.23}$$

$$\varphi = \omega + \sum_{j=2}^{8} C_{2j-1}\, \omega^{-2j+1} . \tag{5.24}$$

To determine ω_{2n-1}, ω_{2n}, one has the equation

$$\sin^2 \tfrac{1}{2}\pi\varphi = O(\omega^{-16}), \tag{5.25}$$

and hence,

$$\varphi = 2n + O(\omega^{-8}). \tag{5.26}$$

Thus

$$\omega + C_3\omega^{-3} + C_5\omega^{-5} + C_7\omega^{-7} = 2n + O(\omega^{-8}). \tag{5.27}$$

Let

$$\omega = 2n + \varepsilon. \tag{5.28}$$

Then

$$\varepsilon + C_3(2n+\varepsilon)^{-3} + C_5(2n+\varepsilon)^{-5} + C_7(2n+\varepsilon)^{-7} = O(n^{-8}). \tag{5.29}$$

Equation (5.29) now yields

$$\varepsilon = -\frac{C_3}{8n^3} - \frac{C_5}{32n^5} - \frac{3C_3{}^2+C_7}{128n^7} + O(n^{-8}). \tag{5.30}$$

One now obtains

$$\lambda = 4n^2 - \frac{C_3}{2n^2} - \frac{C_5}{8n^4} - \frac{5C_3{}^2+2C_7}{64n^6} + O(n^{-7}) . \tag{5.31}$$

Using the explicit values of C_3, C_5, C_7 as given in Equation (4.81), one arrives at the asymptotic expansions given in the theorem.

To determine ω'_{2n-1}, ω'_{2n}, one has the equation

$$\cos^2 \tfrac{1}{2}\pi\varphi \quad = O(\omega^{-16}), \tag{5.32}$$

and hence, $\varphi = 2n - 1 + O(\omega^{-8})$. (5.33)

Let $\omega = 2n - 1 + \varepsilon$. (5.34)

Then one obtains

$$\varepsilon = -\frac{C_3}{(2n-1)^3} - \frac{C_5}{(2n-1)^5} - \frac{3C_3{}^2 + C_7}{(2n-1)^7} + O(n^{-8}). \tag{5.35}$$

One now obtains

$$\lambda' = (2n-1)^2 - \frac{2C_3}{(2n-1)^2} - \frac{2C_5}{(2n-1)^4} - \frac{5C_3{}^2 + 2C_7}{(2n-1)^6} + O(n^{-7}). \tag{5.36}$$

Using the explicit values of C_3, C_5, C_7, one arrives at the asymptotic expansions given in the Theorem.

VI. Conditions that $g(x)$ be of Period $\pi/2$

In the present analysis, the differential Equation (1.1) will be replaced by the family of equations

$$y''(x) + \{\lambda + g(x,\theta)\}y(x) = 0, \tag{6.1}$$

in which

$$g(x,\theta) = \sum_{r=-\infty}^{\infty} e^{i2\theta r}\, g_r\, e^{i2rx}\,. \tag{6.2}$$

The function $g(x)$ is given by $\theta = 0$. It is clear, from $|g(x)| \leq M$ and the periodicity of $g(x)$, that $|g(x,\theta)| \leq M$ uniformly in x and θ. Consider the sequences $\{U_n\}_0^\infty$, $\{V_n\}_0^\infty$ defined by

$$U_n(x) = -1/\omega \sin \int_0^x \omega(x-\xi)g(\xi)U_{n-1}(\xi)d\xi, \quad U_0(x) = \cos\omega x, \tag{6.3}$$

$$V_n(x) = -1/\omega \int_0^x \sin\omega(x-\xi)g(\xi)V_{n-1}(\xi)d\xi, \quad V_0(x) = \sin\omega x/\omega, \tag{6.4}$$

then[1]

$$y_1(x) = \sum_{k=0}^{\infty} U_k(x), \tag{6.5}$$

$$y_2(x) = \sum_{k=0}^{\infty} V_k(x), \tag{6.6}$$

and

$$\Delta(\omega^2) = \sum_{k=0}^{\infty} \Delta_k(\omega^2) \tag{6.7}$$

in which

$$\Delta_k(\omega^2) = U_k(\pi) + V'_k(\pi). \tag{6.8}$$

A theorem of Magnus[1,2] shows that

$$\Delta(\omega^2) = \sum_{k=0}^{n} \Delta_k(\omega^2) + O(|\omega|^{-n-1}e^{\pi|\mu|}). \tag{6.9}$$

In particular,

$$\Delta(\omega^2) = 2 \cos \pi\omega + O(|\omega|^{-2} e^{\pi|\mu|}). \tag{6.10}$$

It will be of interest to compare $\Delta(\omega^2)$ with the function $\bar{\Delta}(\omega^2)$ defined by

$$\bar{\Delta}(\omega^2) = y_1(\pi/2) + y'_2(\pi/2). \tag{6.11}$$

Paralleling the derivation of Equations (6.9) and (6.10), one arrives at similar results for $\bar{\Delta}(\omega^2)$. In particular,

$$\bar{\Delta}_0(\omega^2) = 2 \cos \tfrac{1}{2}\pi \ \omega, \tag{6.12}$$

$$\bar{\Delta}_1(\omega^2) = \frac{\sin \frac{1}{2}\pi \omega}{i\omega} \sum_{r=-\infty}^{\infty} e^{i2\theta(2r+1)} \frac{g_{2r+1}}{2r+1}, \tag{6.13}$$

and

$$\bar{\Delta}(\omega^2) = 2 \cos \tfrac{1}{2}\pi \omega + \frac{\sin \frac{1}{2}\pi \omega}{i\omega} \sum_{r=-\infty}^{\infty} e^{i2\theta(2r+1)} \frac{g_{2r+1}}{2r+1} \tag{6.14}$$

$$+ O(|\omega|^{-2} e^{\pi|\mu|}).$$

Define the function $D(\omega^2,\theta)$ by

$$D(\omega^2) = \Delta(\omega^2,\theta) + 2 - \bar{\Delta}(\omega^2,\theta)^2. \tag{6.15}$$

The following theorem will now be proved.

Theorem XIX. $D(\omega^2,\theta) = o(|\omega|^{-1}) \Leftrightarrow g_{2r+1} = 0(-\infty < r < \infty)$ ω real. The estimate holding uniformly in θ.

Proof. If $g_{2r+1} = 0(-\infty < r < \infty)$ then $g(x)$ is of period $\pi/2$ and $y_1(x + \pi/2)$, $y_2(x + \pi/2)$ are solutions of Hill's equation.

Thus

$$\begin{bmatrix} y_1(x + \pi/2) \\ y_2(x + \pi/2) \end{bmatrix} = \begin{bmatrix} y_1(\pi/2), & y'_1(\pi/2) \\ y_2(\pi/2), & y'_2(\pi/2) \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} \tag{6.16}$$

and

$$\begin{bmatrix} y_1(x+\pi) \\ y_2(x+\pi) \end{bmatrix} = \begin{bmatrix} y_1(\pi/2), & y'_1(\pi/2) \\ y_2(\pi/2), & y'_2(\pi/2) \end{bmatrix}^2 \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} . \tag{6.17}$$

Since

$$\begin{bmatrix} y_1(x+\pi) \\ y_2(x+\pi) \end{bmatrix} = \begin{bmatrix} y_1(\pi), & y'_1(\pi) \\ y_2(\pi), & y'_2(\pi) \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} . \tag{6.18}$$

One has

$$\begin{bmatrix} y_1(\pi/2), & y'_1(\pi/2) \\ y_2(\pi/2), & y'_2(\pi/2) \end{bmatrix}^2 = \begin{bmatrix} y_1(\pi), & y'_1(\pi) \\ y_2(\pi), & y'_2(\pi) \end{bmatrix} . \tag{6.19}$$

Thus

$$\Delta(\omega^2,\theta) + 2 = y_1(\pi/2)^2 + y'_2(\pi/2)^2 + 2y'_1(\pi/2)y_2(\pi/2) + 2. \tag{6.20}$$

From the Wronskian

$$\begin{vmatrix} y_1(x), & y_2(x) \\ y'_1(x), & y'_2(x) \end{vmatrix} = y_1(x)y'_2(x) - y'_1(x)\, y_2(x) = 1 , \tag{6.21}$$

one obtains $2y'_1(\pi/2)y_2(\pi/2) = 2y_1(\pi/2)y'_2(\pi/2) - 2$. (6.22)

Hence $\Delta(\omega^2,\theta) + 2 = [y_1(\pi/2) + y'_2(\pi/2)]^2 = \bar{\Delta}(\omega^2,\theta)^2$. (6.33)

Thus if $g_{2r+1} = 0(-\infty < r < \infty)$, then $D(\omega^2,\theta) = 0 = o(|\omega|^{-1})$. (6.34)

Equations (6.10), (6.14), and (6.15) yield

$$D(\omega^2,\theta) = -\frac{2\sin\pi\omega}{i\omega} \sum_{r=-\infty}^{\infty} e^{i2\theta(2r+1)} \frac{g_{2r+1}}{2r+1} + O(\omega^{-2}) \tag{6.25}$$

in which ω is real. The condition $D(\omega^2,\theta) = o(|\omega|^{-1})$ implies that

$$\sum_{r=-\infty}^{\infty} e^{i2\theta(2r+1)} \frac{g_{2r+1}}{2r+1} = 0 \tag{6.26}$$

for all θ. Hence, by the uniqueness theorem for Fourier series, $g_{2r+1} = 0 (-\infty < r < \infty)$.

The next theorem presents an interpolatory property of $D(\omega^2)$ which is interesting because it follows from little quantitative information on ω_j.

Theorem XX. All zeros ω_j of $\Delta(\omega^2) + 2$ are double and $D(\omega_j^{\,2}) = 0(-\infty < j < \infty)$ $\Longrightarrow D(\omega^2) \equiv 0$.

Proof. Consider the integrals in the ζ-plane ($\zeta = \xi + i\eta$, ξ,η real)

$$I^{(j)} = \frac{1}{2\pi i}\int_{C_j} \frac{D(\zeta^2)}{(\zeta-\omega)[\Delta(\zeta^2)+2]}\, d\zeta \,. \tag{6.27}$$

The paths C_j are squares to be defined below. If

$$\lim_{j\to\infty} I^{(j)} = 0, \tag{6.28}$$

then one has, on application of the calculus of residues,

$$D(\omega^2) = -\sum_{j=-\infty}^{\infty} \frac{\Delta(\omega^2)+2}{2\omega_j^{\,2}\Delta''(\omega_j^{\,2})}\left[\frac{\bar{\Delta}(\omega_j^{\,2})^2}{(\omega-\omega_j)^2} + \frac{2\omega_j\bar{\Delta}(\omega_j^{\,2})\bar{\Delta}'(\omega_j^{\,2})}{\omega-\omega_j}\right], \tag{6.29}$$

in which the series converges uniformly in every closed bounded domain of the ω-plane. The theorem follows directly from Equation (6.29). It is only necessary, therefore, to establish the validity of Equation (6.28).

From Equation (6.10) one has

$$\Delta(\omega^2) + 2 = 4\cos^2\frac{\pi}{2}\omega + O(\omega^{-2}) \tag{6.30}$$

for real ω. Thus,

$$\cos^2 \frac{\pi}{2} \omega_j = O(\omega_j^{-2}) . \tag{6.31}$$

Let

$$\omega_j = 2j - 1 + \varepsilon_j. \tag{6.32}$$

Then

$$\varepsilon_j = O(|j|^{-1}), \tag{6.33}$$

and hence

$$\omega_j = 2j - 1 + O(|j|^{-1}) . \tag{6.34}$$

A considerably sharper result is available in Theorem XVIII; however, no more than is given in Equation (6.34) will be needed. It therefore presents independent interest that the crude estimate in Equation (6.30) suffices for the investigation. Define Ω_j by

$$\Omega_j = \frac{\omega_j + \omega_{j+1}}{2} = 2j + O(|j|^{-1}), \tag{6.35}$$

then the corners of the squares C_j are given by $\pm \Omega_j \pm i\Omega_j$. Let

$$I^{(j)} = I_1 + I_2 + I_3 + I_4 \tag{6.36}$$

in which

$$I_1 = \frac{1}{2\pi i} \int_{-\Omega_j}^{\Omega_j} \frac{D(\zeta^2)}{(\zeta - \omega)[\Delta(\zeta^2) + 2]} \, i d\eta, \ \xi = \Omega_j, \tag{6.37}$$

$$I_2 = \frac{1}{2\pi i} \int_{\Omega_j}^{-\Omega_j} \frac{D(\zeta^2)}{(\zeta - \omega)[\Delta(\zeta^2) + 2]} \, d\xi, \ \eta = \Omega_j, \tag{6.38}$$

$$I_3 = \frac{1}{2\pi i} \int_{\Omega_j}^{-\Omega_j} \frac{D(\zeta^2)}{(\zeta - \omega)[\Delta(\zeta^2) + 2]} \, i d\eta, \xi = -\Omega_j, \tag{6.39}$$

$$I_4 = \frac{1}{2\pi i} \int_{-\Omega_j}^{\Omega_j} \frac{D(\zeta^2)}{(\zeta - \omega)[\Delta(\zeta^2) + 2]} \, d\xi, \ \eta = -\Omega_j. \tag{6.40}$$

Equation (6.25) shows that

$$D(\omega^2) = O(|\omega|^{-1} e^{\pi|\mu|}). \tag{6.41}$$

One has

$$|I_1| \leq \frac{1}{2\pi} \int_{-\Omega_j}^{\Omega_j} \frac{D(\zeta^2)}{|\Omega_j + i\eta - \omega| |\Delta(\zeta^2) + 2|} \, d\eta \, . \tag{6.42}$$

Since

$$|\Omega_j + i\eta - \omega| \geq |\Omega_j| - |\omega| \geq 2j - O(j^{-1}) - |\omega| \; (j > 0), \tag{6.43}$$

one may choose j so large that $2j - O(j^{-1}) - |\omega| > 0$, then

$$I_1 = O\left\{ \frac{1}{j} \int_{-\Omega_j}^{\Omega_j} \frac{|D(\zeta^2)|}{|\Delta(\zeta^2)+2|} \, d\eta \right\} . \tag{6.44}$$

One has, from Equation (6.41),

$$D(\zeta^2) = O(|\zeta|^{-1} e^{\pi|\eta|}), \tag{6.45}$$

hence,

$$I_1 = O\left\{ \frac{1}{j^2} \int_{-\Omega_j}^{\Omega_j} \frac{e^{\pi|\eta|}}{|\Delta(\zeta^2)+2|} \, d\eta \right\} . \tag{6.46}$$

From Equations (6.10) and (6.35) one has

$$|\Delta(\zeta^2) + 2| \geq O(e^{\pi|\eta|}) \tag{6.47}$$

for j large enough, hence

$$I_1 = O(j^{-1}) \tag{6.48}$$

and $\lim\limits_{j\to\infty} I_1 = 0$.

The integral I_3 follows exactly the same analysis so that one has also

$$I_3 = O(j^{-1}) \quad \text{and} \quad \lim_{j\to\infty} I_3 = 0.$$

For the integral I_2 one has

$$|I_2| \leq \frac{1}{2\pi} \int_{-\Omega_j}^{\Omega_j} \frac{|D(\zeta^2)|}{|\zeta-\omega|\,|\Delta(\zeta^2)+2|} \, d\xi \; . \tag{6.49}$$

Since

$$|\xi + i\Omega_j - \omega| \geq |\Omega_j| - |\omega| > 2j - O(j^{-1}) - |\omega| \; (j > 0), \tag{6.50}$$

one may choose j so large that $2j-O(j^{-1})-|\omega| > 0$, then

$$I_2 = O\{ \frac{1}{j} \int_{-\Omega_j}^{\Omega_j} \frac{|D(\zeta^2)|}{|\Delta(\zeta^2)+2|} \, d\xi \} \; . \tag{6.51}$$

Using Equation (6.45), one obtains

$$I_2 = O\{ \frac{e^{2\pi j}}{j^2} \int_{-\Omega_j}^{\Omega_j} \frac{1}{|\Delta(\zeta^2)+2|} \, d\xi \} \; . \tag{6.52}$$

Since

$$|\cos \frac{\pi}{2} \omega_j| \geq \frac{e^{\pi j} - e^{-\pi j}}{2} \{1 + O(j^{-1})\}, \tag{6.53}$$

one has

$$|\Delta(\zeta^2) + 2| \geq O(e^{2\pi j}) \tag{6.54}$$

for sufficiently large j uniformly in ξ, hence

$$I_2 = O(j^{-1}), \tag{6.55}$$

and

$$\lim_{j\to\infty} I_2 = 0.$$

Exactly the same analysis applies to I_4 so that $I_4 = O(j^{-1})$ and $\lim_{j\to\infty} I_4 = 0$. The theorem is now established.

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